

# Package ‘stokes’

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**Type** Package

**Title** The Exterior Calculus

**Version** 1.2-3

**Depends** R (>= 4.1.0)

**Suggests** knitr, Deriv, testthat, markdown, rmarkdown, quadform,  
magrittr, covr

**VignetteBuilder** knitr

**Imports** permutations (>= 1.1-2), partitions, methods, disordR (>= 0.9-7), spray (>= 1.0-26)

**Maintainer** Robin K. S. Hankin <hankin.robin@gmail.com>

**Description** Provides functionality for working with tensors, alternating forms, wedge products, Stokes's theorem, and related concepts from the exterior calculus. Uses 'disordR' discipline (Hankin, 2022, <doi:10.48550/arXiv.2210.03856>). The canonical reference would be M. Spivak (1965, ISBN:0-8053-9021-9) ``Calculus on Manifolds''. To cite the package in publications please use Hankin (2022) <doi:10.48550/arXiv.2210.17008>.

**License** GPL-2

**LazyData** yes

**URL** <https://github.com/RobinHankin/stokes>,  
<https://robinhankin.github.io/stokes/>

**BugReports** <https://github.com/RobinHankin/stokes/issues>

**NeedsCompilation** no

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stokes-package

*The Exterior Calculus*

---

### Description

Provides functionality for working with tensors, alternating forms, wedge products, Stokes's theorem, and related concepts from the exterior calculus. Uses 'disordR' discipline (Hankin, 2022, <doi:10.48550/arXiv.2210.03856>). The canonical reference would be M. Spivak (1965, ISBN:0-8053-9021-9) "Calculus on Manifolds". To cite the package in publications please use Hankin (2022) <doi:10.48550/arXiv.2210.17008>.

**Details**

The DESCRIPTION file:

```

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Version:     1.2-3
Depends:     R (>= 4.1.0)
Suggests:    knitr, Deriv, testthat, markdown, rmarkdown, quadform, magrittr, covr
VignetteBuilder: knitr
Imports:     permutations (>= 1.1-2), partitions, methods, disordR (>= 0.9-7), spray (>= 1.0-26)
Authors@R:   person( given=c("Robin", "K. S."), family="Hankin", role = c("aut","cre"), email="hankin.robin@gmail.com)
Maintainer:  Robin K. S. Hankin <hankin.robin@gmail.com>
Description: Provides functionality for working with tensors, alternating forms, wedge products, Stokes's theorem, and
License:     GPL-2
LazyData:    yes
URL:         https://github.com/RobinHankin/stokes, https://robinhankin.github.io/stokes/
BugReports:  https://github.com/RobinHankin/stokes/issues
Author:      Robin K. S. Hankin [aut, cre] (<https://orcid.org/0000-0001-5982-0415>)

```

Index of help topics:

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Ops.kform	Arithmetic Ops Group Methods for 'kform' and 'ktensor' objects
as.1form	Coerce vectors to 1-forms
coeffs	Extract and manipulate coefficients
consolidate	Various low-level helper functions
contract	Contractions of k-forms
dovs	Dimension of the underlying vector space
dx	Elementary forms in three-dimensional space
ex	Basis vectors in three-dimensional space
hodge	Hodge star operator
inner	Inner product operator
issmall	Is a form zero to within numerical precision?
keep	Keep or drop variables
kform	k-forms
kinner	Inner product of two kforms
ktensor	k-tensors
phi	Elementary tensors
print.stokes	Print methods for k-tensors and k-forms
rform	Random kforms and ktensors
scalar	Scalars and losing attributes
stokes-package	The Exterior Calculus
summary.stokes	Summaries of tensors and alternating forms
symbolic	Symbolic form
tensorprod	Tensor products of k-tensors

transform	Linear transforms of k-forms
vector_cross_product	The Vector cross product
volume	The volume element
wedge	Wedge products
zap	Zap small values in k-forms and k-tensors
zero	Zero tensors and zero forms

Generally in the package, arguments that are  $k$ -forms are denoted  $K$ ,  $k$ -tensors by  $U$ , and spray objects by  $S$ . Multilinear maps (which may be either  $k$ -forms or  $k$ -tensors) are denoted by  $M$ .

### Author(s)

Robin K. S. Hankin [aut, cre] (<<https://orcid.org/0000-0001-5982-0415>>)

Maintainer: Robin K. S. Hankin <[hankin.robin@gmail.com](mailto:hankin.robin@gmail.com)>

### References

- M. Spivak 1971. *Calculus on manifolds*, Addison-Wesley.
- R. K. S. Hankin 2022. “Disordered vectors in R: introducing the **disordR** package.” <https://arxiv.org/abs/2210.03856>.
- R. K. S. Hankin 2022. “Sparse arrays in R: the **spray** package. <https://arxiv.org/abs/2210.03856>.”

### See Also

[spray](#)

### Examples

```
## Some k-tensors:
U1 <- as.ktensor(matrix(1:15,5,3))
U2 <- as.ktensor(cbind(1:3,2:4),1:3)

## Coerce a tensor to functional form, here mapping V^3 -> R (here V=R^15):
as.function(U1)(matrix(rnorm(45),15,3))

## Tensor product is tensorprod() or %X%:
U1 %X% U2

## A k-form is an alternating k-tensor:
K1 <- as.kform(cbind(1:5,2:6),rnorm(5))
K2 <- kform_general(3:6,2,1:6)
K3 <- rform(9,3,9,runif(9))

## The distributive law is true
(K1 + K2) ^ K3 == K1 ^ K3 + K2 ^ K3 # TRUE to numerical precision

## Wedge product is associative (non-trivial):
```

```

(K1 ^ K2) ^ K3
K1 ^ (K2 ^ K3)

## k-forms can be coerced to a function and wedge product:
f <- as.function(K1 ^ K2 ^ K3)

## E is a a random point in V^k:
E <- matrix(rnorm(63),9,7)

## f() is alternating:
f(E)
f(E[,7:1])

## The package blurs the distinction between symbolic and numeric computing:
dx <- as.kform(1)
dy <- as.kform(2)
dz <- as.kform(3)

dx ^ dy ^ dz

K3 ^ dx ^ dy ^ dz

```

---

Alt

*Alternating multilinear forms*


---

## Description

Converts a  $k$ -tensor to alternating form

## Usage

```
Alt(S, give_kform)
```

## Arguments

S	A multilinear form, an object of class ktensor
give_kform	Boolean, with default FALSE meaning to return an alternating $k$ -tensor [that is, an object of class ktensor that happens to be alternating] and TRUE meaning to return a $k$ -form [that is, an object of class kform]

## Details

Given a  $k$ -tensor  $T$ , we have

$$\text{Alt}(T)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \cdot T(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

Thus for example if  $k = 3$ :

$$\text{Alt}(T)(v_1, v_2, v_3) = \frac{1}{6} \begin{pmatrix} +T(v_1, v_2, v_3) & -T(v_1, v_3, v_2) \\ -T(v_2, v_1, v_3) & +T(v_2, v_3, v_1) \\ +T(v_3, v_1, v_2) & -T(v_3, v_2, v_1) \end{pmatrix}$$

and it is reasonably easy to see that  $\text{Alt}(T)$  is alternating, in the sense that

$$\text{Alt}(T)(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\text{Alt}(T)(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$$

Function `Alt()` is intended to take and return an object of class `ktensor`; but if given a `kform` object, it just returns its argument unchanged.

A short vignette is provided with the package: `type vignette("Alt")` at the commandline.

### Value

Returns an alternating  $k$ -tensor. To work with  $k$ -forms, which are a much more efficient representation of alternating tensors, use `as.kform()`.

### Author(s)

Robin K. S. Hankin

### See Also

[kform](#)

### Examples

```
(X <- ktensor(spray(rbind(1:3),6)))
Alt(X)
Alt(X,give_kform=TRUE)

S <- as.ktensor(expand.grid(1:3,1:3),rnorm(9))
S
Alt(S)

issmall(Alt(S) - Alt(Alt(S))) # should be TRUE; Alt() is idempotent

a <- rtensor()
V <- matrix(rnorm(21),nrow=3)
LHS <- as.function(Alt(a))(V)
RHS <- as.function(Alt(a,give_kform=TRUE))(V)
c(LHS=LHS,RHS=RHS,diff=LHS-RHS)
```

as.1form

*Coerce vectors to 1-forms***Description**

Given a vector, return the corresponding 1-form; the exterior derivative of a 0-form (that is, a scalar function). Function `grad()` is a synonym.

**Usage**

```
as.1form(v)
grad(v)
```

**Arguments**

`v` A vector with element  $i$  being  $\partial f / \partial x_i$

**Details**

The exterior derivative of a  $k$ -form  $\phi$  is a  $(k + 1)$ -form  $d\phi$  given by

$$d\phi(P_{\mathbf{x}}(\mathbf{v}_1, \dots, \mathbf{v}_{k+1})) = \lim_{h \rightarrow 0} \frac{1}{h^{k+1}} \int_{\partial P_{\mathbf{x}}(h\mathbf{v}_1, \dots, h\mathbf{v}_{k+1})} \phi$$

We can use the facts that

$$d(f dx_{i_1} \wedge \dots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

and

$$df = \sum_{j=1}^n (D_j f) dx_j$$

to calculate differentials of general  $k$ -forms. Specifically, if

$$\phi = \sum_{1 \leq i_1 < \dots < i_k \leq n} a_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

then

$$d\phi = \sum_{1 \leq i_1 < \dots < i_k \leq n} \left[ \sum_{j=1}^n D_j a_{i_1 \dots i_k} dx_j \right] \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

The entry in square brackets is given by `grad()`. See the examples for appropriate R idiom.

**Value**

A one-form

**Author(s)**

Robin K. S. Hankin

**See Also**

[kform](#)

**Examples**

```
as.1form(1:9) # note ordering of terms
```

```
as.1form(rnorm(20))
```

```
grad(c(4,7)) ^ grad(1:4)
```

---

coeffs

*Extract and manipulate coefficients*

---

**Description**

Extract and manipulate coefficients of ktensor and kform objects; this using the methods of the **spray** package.

Functions `as.spray()` and `nterms()` are imported from **spray**.

**Details**

To see the coefficients of a kform or ktensor object, use `coeffs()`, which returns a `disord` object (this is actually `spray::coeffs()`). Replacement methods also use the methods of the **spray** package. Note that **disordR** discipline is enforced.

Experimental functionality for “pure” extraction and replacement is provided, following **spray** version 1.0-25 or above. Thus idiom such as `a[abs(coeffs(a)) > 0.1]` or indeed `a[coeffs(a) < 1] <- 0` should work as expected.

**Author(s)**

Robin K. S. Hankin



**Examples**

```
(a <- kform_general(5,2,1:10))
coeffs(a) # a disord object
coeffs(a)[coeffs(a)%%2==1] <- 100 # replace every odd coeff with 100
a

coeffs(a*0)

a <- rform()
a[coeffs(a) < 5] # experimental
a[coeffs(a) > 3] <- 99 # experimental
```

---

consolidate	<i>Various low-level helper functions</i>
-------------	---

---

**Description**

Various low-level helper functions used in `Alt()` and `kform()`

**Usage**

```
consolidate(S)
kill_trivial_rows(S)
include_perms(S)
kform_to_ktensor(S)
```

**Arguments**

`S`                      Object of class `spray`

**Details**

Low-level helper functions.

- Function `consolidate()` takes a `spray` object, and combines any rows that are identical up to a permutation, respecting the sign of the permutation
- Function `kill_trivial_rows()` takes a `spray` object and deletes any rows with a repeated entry (which have  $k$ -forms identically zero)
- Function `include_perms()` replaces each row of a `spray` object with all its permutations, respecting the sign of the permutation
- Function `ktensor_to_kform()` coerces a  $k$ -form to a  $k$ -tensor

**Value**

The functions documented here all return a `spray` object.

**Author(s)**

Robin K. S. Hankin

**See Also**[ktensor](#), [kform](#), [Alt](#)**Examples**

```
(S <- spray(matrix(c(1,1,2,2,1,3,3,1,3,5),ncol=2,byrow=TRUE),1:5))

kill_trivial_rows(S) # (rows 1 and 3 killed, repeated entries)
consolidate(S)      # (merges rows 2 and 4)
include_perms(S)   # returns a spray object, not alternating tensor.
```

contract

*Contractions of  $k$ -forms***Description**

A contraction is a natural linear map from  $k$ -forms to  $k - 1$ -forms.

**Usage**

```
contract(K, v, lose=TRUE)
contract_elementary(o, v)
```

**Arguments**

K	A $k$ -form
o	Integer-valued vector corresponding to one row of an index matrix
lose	Boolean, with default TRUE meaning to coerce a 0-form to a scalar and FALSE meaning to return the formal 0-form
v	A vector; in function <code>contract()</code> , if a matrix, interpret each column as a vector to contract with

**Details**

Given a  $k$ -form  $\phi$  and a vector  $\mathbf{v}$ , the *contraction*  $\phi_{\mathbf{v}}$  of  $\phi$  and  $\mathbf{v}$  is a  $k - 1$ -form with

$$\phi_{\mathbf{v}}(\mathbf{v}^1, \dots, \mathbf{v}^{k-1}) = \phi(\mathbf{v}, \mathbf{v}^1, \dots, \mathbf{v}^{k-1})$$

provided  $k > 1$ ; if  $k = 1$  we specify  $\phi_{\mathbf{v}} = \phi(\mathbf{v})$ .

Function `contract_elementary()` is a low-level helper function that translates elementary  $k$ -forms with coefficient 1 (in the form of an integer vector corresponding to one row of an index matrix) into its contraction with  $\mathbf{v}$ .

There is an extensive vignette in the package, `vignette("contract")`.

**Value**

Returns an object of class kform.

**Author(s)**

Robin K. S. Hankin

**References**

Steven H. Weintraub 2014. “Differential forms: theory and practice”, Elsevier (Definition 2.2.23, chapter 2, page 77).

**See Also**

[wedge,lose](#)

**Examples**

```
contract(as.kform(1:5),1:8)
contract(as.kform(1),3) # 0-form

contract_elementary(c(1,2,5),c(1,2,10,11,71))

## Now some verification [takes ~10s to run]:
#o <- kform(spray(t(replicate(2, sample(9,4))), runif(2)))
#V <- matrix(rnorm(36),ncol=4)
#jj <- c(
# as.function(o)(V),
# as.function(contract(o,V[,1,drop=TRUE]))(V[,-1]), # scalar
# as.function(contract(o,V[,1:2]))(V[,-(1:2),drop=FALSE]),
# as.function(contract(o,V[,1:3]))(V[,-(1:3),drop=FALSE]),
# as.function(contract(o,V[,1:4],lose=FALSE))(V[,-(1:4),drop=FALSE])
#)

#print(jj)
#max(jj) - min(jj) # zero to numerical precision
```

**Description**

A  $k$ -form  $\omega \in \Lambda^k(V)$  maps  $V^k$  to the reals, where  $V = \mathbb{R}^n$ . Function `dovs()` returns  $n$ , the dimensionality of the underlying vector space. The function itself is almost trivial, returning the maximum of the index matrix.

Special dispensation is given for zero-forms and zero tensors, which return zero.

Vignette `dovs` provides more discussion.

**Usage**

```
dovs(K)
```

**Arguments**

K                    A  $k$ -form or  $k$ -tensor

**Value**

Returns a non-negative integer

**Author(s)**

Robin K. S. Hankin

**Examples**

```
dovs(rform())
table(replicate(20, dovs(rform(3))))
```

dx

*Elementary forms in three-dimensional space*

**Description**

Objects `dx`, `dy` and `dz` are the three elementary one-forms on three-dimensional space. These objects can be generated by running script ‘`vignettes/dx.Rmd`’, which includes some further discussion and technical documentation and creates file ‘`dx.rda`’ which resides in the `data/` directory.

The default print method is a little opaque for these objects. To print them more intuitively, use

```
options(kform_symbolic_print = "dx")
```

which is documented at `print.Rd`.

**Usage**

```
data(dx)
```

**Details**

See vignettes dx and ex for an extended discussion; a use-case is given in `vector_cross_product`.

**Author(s)**

Robin K. S. Hankin

**References**

- M. Spivak 1971. *Calculus on manifolds*, Addison-Wesley

**See Also**

[d,print.kform](#)

**Examples**

```
dx
hodge(dx)
hodge(dx, 3)

dx # default print method, not particularly intelligible
options(kform_symbolic_print = 'dx') # shows dx dy dz
dx
dx^dz
hodge(dx, 3)

as.function(dx)(ex)

options(kform_symbolic_print = NULL) # revert to default
```

---

ex

*Basis vectors in three-dimensional space*


---

**Description**

Objects `ex`, `ey` and `ez` are the three elementary one-forms on three-dimensional space, sometimes denoted  $(e_x, e_y, e_z)$ . These objects can be generated by running script ‘`vignettes/ex.Rmd`’, which includes some further discussion and technical documentation and creates file ‘`exeyez.rda`’ which resides in the `data/` directory.

**Details**

See vignettes dx and ex for an extended discussion; a use-case is given in `vector_cross_product`.

**Author(s)**

Robin K. S. Hankin

**References**

- M. Spivak 1971. *Calculus on manifolds*, Addison-Wesley

**See Also**

[d,print.kform](#)

**Examples**

```
as.function(dx)(ex)

(X <- as.kform(matrix(1:12,nrow=4),c(1,2,7,11)))
as.function(X)(cbind(e(2,12),e(6,12),e(10,12)))
```

---

hodge

*Hodge star operator*

---

**Description**

Given a  $k$ -form, return its Hodge dual

**Usage**

```
hodge(K, n=dovs(K), g, lose=TRUE)
```

**Arguments**

K	Object of class kform
n	Dimensionality of space, defaulting to the largest element of the index
g	Diagonal of the metric tensor, with missing default being the standard metric of the identity matrix. Currently, only entries of $\pm 1$ are accepted
lose	Boolean, with default TRUE meaning to coerce to a scalar if appropriate

**Value**

Given a  $k$ -form, in an  $n$ -dimensional space, return a  $(n - k)$ -form.

**Note**

Most authors write the Hodge dual of  $\psi$  as  $*\psi$  or  $\star\psi$ , but Weintraub uses  $\psi^*$ .

**Author(s)**

Robin K. S. Hankin

**See Also**[wedge](#)**Examples**

```
(o <- kform_general(5,2,1:10))
hodge(o)
o == hodge(hodge(o))

Faraday <- kform_general(4,2,runif(6)) # Faraday electromagnetic tensor
mink <- c(-1,1,1,1) # Minkowski metric
hodge(Faraday,g=mink)

Faraday == Faraday |>
  hodge(g=mink) |>
  hodge(g=mink) |>
  hodge(g=mink) |>
  hodge(g=mink)

hodge(dx,3) == dy^dz

## Some edge-cases:
hodge(scalar(1),2)
hodge(zeroform(5),9)
hodge(volume(5))
hodge(volume(5),lose=TRUE)
hodge(scalar(7),n=9)
```

---

inner

*Inner product operator*

---

**Description**

The inner product

**Usage**

inner(M)

**Arguments**

`M` square matrix

**Details**

The inner product of two vectors  $\mathbf{x}$  and  $\mathbf{y}$  is usually written  $\langle \mathbf{x}, \mathbf{y} \rangle$  or  $\mathbf{x} \cdot \mathbf{y}$ , but the most general form would be  $\mathbf{x}^T M \mathbf{y}$  where  $M$  is a matrix. Noting that inner products are multilinear, that is  $\langle \mathbf{x}, a\mathbf{y} + b\mathbf{z} \rangle = a \langle \mathbf{x}, \mathbf{y} \rangle + b \langle \mathbf{x}, \mathbf{z} \rangle$  and  $\langle a\mathbf{x} + b\mathbf{y}, \mathbf{z} \rangle = a \langle \mathbf{x}, \mathbf{z} \rangle + b \langle \mathbf{y}, \mathbf{z} \rangle$ , we see that the inner product is indeed a multilinear map, that is, a tensor.

Given a square matrix  $M$ , function `inner(M)` returns the 2-form that maps  $\mathbf{x}, \mathbf{y}$  to  $\mathbf{x}^T M \mathbf{y}$ . Non-square matrices are effectively padded with zeros.

A short vignette is provided with the package: `type vignette("inner")` at the commandline.

**Value**

Returns a  $k$ -tensor, an inner product

**Author(s)**

Robin K. S. Hankin

**See Also**

[kform](#)

**Examples**

```
inner(diag(7))
inner(matrix(1:9,3,3))

## Compare the following two:
Alt(inner(matrix(1:9,3,3))) # An alternating k tensor
as.kform(inner(matrix(1:9,3,3))) # Same thing coerced to a kform

f <- as.function(inner(diag(7)))
X <- matrix(rnorm(14),ncol=2) # random element of (R^7)^2
f(X) - sum(X[,1]*X[,2]) # zero to numerical precision

## verify positive-definiteness:
g <- as.function(inner(crossprod(matrix(rnorm(56),8,7))))
stopifnot(g(kronecker(rnorm(7),t(c(1,1))))>0)
```



---

issmall	<i>Is a form zero to within numerical precision?</i>
---------	--

---

**Description**

Given a  $k$ -form, return TRUE if it is “small”

**Usage**

```
issmall(M, tol=1e-8)
```

**Arguments**

M	Object of class kform or ktensor
tol	Small tolerance, defaulting to 1e-8

**Value**

Returns a logical

**Author(s)**

Robin K. S. Hankin

**Examples**

```
o <- kform_general(3,2,runif(3))
M <- matrix(rnorm(9),3,3)

discrepancy <- o - pullback(pullback(o,M),solve(M))

discrepancy # print method might imply coefficients are zeros

issmall(discrepancy) # should be TRUE
is.zero(discrepancy) # might be FALSE
```

---

keep	<i>Keep or drop variables</i>
------	-------------------------------

---

**Description**

Keep or drop variables

**Usage**

```
keep(K, yes)
discard(K, no)
```

**Arguments**

K                    Object of class kform  
 yes, no            Specification of dimensions to either keep (yes) or discard (no)

**Details**

Function `keep(omega, yes)` keeps the terms specified and `discard(omega, no)` discards the terms specified. It is not clear to me what these functions mean from a mathematical perspective.

**Value**

The functions documented here all return a kform object.

**Author(s)**

Robin K. S. Hankin

**See Also**

[lose](#)

**Examples**

```
(o <- kform_general(7,3,seq_len(choose(7,3))))
keep(o,1:4) # keeps only terms with dimensions 1-4
discard(o,1:2) # loses any term with a "1" in the index
```

---

kform

*k-forms*

---

**Description**

Functionality for dealing with *k*-forms

**Usage**

```
kform(S)
as.kform(M, coeffs, lose=TRUE)
kform_basis(n, k)
kform_general(W, k, coeffs, lose=TRUE)
is.kform(x)
d(i)
e(i, n)
## S3 method for class 'kform'
as.function(x, ...)
```

**Arguments**

n	Dimension of the vector space $V = \mathbb{R}^n$
i	Integer
k	A $k$ -form maps $V^k$ to $\mathbb{R}$
W	Integer vector of dimensions
M, coeffs	Index matrix and coefficients for a $k$ -form
S	Object of class <code>spray</code>
lose	Boolean, with default TRUE meaning to coerce a 0-form to a scalar and FALSE meaning to return the formal 0-form
x	Object of class <code>kform</code>
...	Further arguments, currently ignored

**Details**

A  $k$ -form is an alternating  $k$ -tensor. In the package,  $k$ -forms are represented as sparse arrays (`spray` objects), but with a class of `c("kform", "spray")`. The constructor function `kform()` takes a `spray` object and returns a `kform` object: it ensures that rows of the index matrix are strictly non-negative integers, have no repeated entries, and are strictly increasing. Function `as.kform()` is more user-friendly.

- `kform()` is the constructor function. It takes a `spray` object and returns a `kform`.
- `as.kform()` also returns a `kform` but is a bit more user-friendly than `kform()`.
- `kform_basis()` is a low-level helper function that returns a matrix whose rows constitute a basis for the vector space  $\Lambda^k(\mathbb{R}^n)$  of  $k$ -forms.
- `kform_general()` returns a `kform` object with terms that span the space of alternating tensors.
- `is.kform()` returns TRUE if its argument is a `kform` object.
- `d()` is an easily-typed synonym for `as.kform()`. The idea is that  $d(1) = dx$ ,  $d(2)=dy$ ,  $d(5)=dx^5$ , etc. Also note that, for example,  $d(1:3)=dx^dy^dz$ , the volume form.

Recall that a  $k$ -tensor is a multilinear map from  $V^k$  to the reals, where  $V = \mathbb{R}^n$  is a vector space. A multilinear  $k$ -tensor  $T$  is *alternating* if it satisfies

$$T(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -T(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$$

In the package, an object of class `kform` is an efficient representation of an alternating tensor.

Function `kform_basis()` is a low-level helper function that returns a matrix whose rows constitute a basis for the vector space  $\Lambda^k(\mathbb{R}^n)$  of  $k$ -forms:

$$\phi = \sum_{1 \leq i_1 < \dots < i_k \leq n} a_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

and indeed we have:

$$a_{i_1 \dots i_k} = \phi(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_k})$$

where  $\mathbf{e}_j$ ,  $1 \leq j \leq k$  is a basis for  $V$ .

**Value**

All functions documented here return a kform object except `as.function.kform()`, which returns a function, and `is.kform()`, which returns a Boolean, and `e()`, which returns a conjugate basis to that of `d()`.

**Note**

Hubbard and Hubbard use the term “*k*-form”, but Spivak does not.

**Author(s)**

Robin K. S. Hankin

**References**

Hubbard and Hubbard; Spivak

**See Also**

[ktensor](#), [lose](#)

**Examples**

```
as.kform(cbind(1:5,2:6),rnorm(5))
kform_general(1:4,2,coeffs=1:6) # used in electromagnetism

K1 <- as.kform(cbind(1:5,2:6),rnorm(5))
K2 <- kform_general(5:8,2,1:6)
K1^K2 # or wedge(K1,K2)

d(1:3)
dx^dy^dz # same thing

d(sample(9)) # coeff is +/-1 depending on even/odd permutation of 1:9

f <- as.function(wedge(K1,K2))
E <- matrix(rnorm(32),8,4)
f(E) + f(E[,c(1,3,2,4)]) # should be zero by alternating property

options(kform_symbolic_print = 'd')
(d(5)+d(7)) ^ (d(2)^d(5) + 6*d(4)^d(7))
options(kform_symbolic_print = NULL) # revert to default
```

kinner

*Inner product of two kforms***Description**

Given two  $k$ -forms  $\alpha$  and  $\beta$ , return the inner product  $\langle \alpha, \beta \rangle$ . Here our underlying vector space  $V$  is  $\mathcal{R}^n$ .

The inner product is a symmetric bilinear form defined in two stages. First, we specify its behaviour on decomposable  $k$ -forms  $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k$  and  $\beta = \beta_1 \wedge \cdots \wedge \beta_k$  as

$$\langle \alpha, \beta \rangle = \det \left( \langle \alpha_i, \beta_j \rangle_{1 \leq i, j \leq k} \right)$$

and secondly, we extend to the whole of  $\Lambda^k(V)$  through linearity.

**Usage**

```
kinner(o1, o2, M)
```

**Arguments**

o1, o2	Objects of class kform
M	Matrix

**Value**

Returns a real number

**Note**

There is a vignette available: type `vignette("kinner")` at the command line.

**Author(s)**

Robin K. S. Hankin

**See Also**

[hodge](#)

**Examples**

```
a <- (2*dx)^(3*dy)
b <- (5*dx)^(7*dy)

kinner(a,b)
det(matrix(c(2*5,0,0,3*7),2,2)) # mathematically identical, slight numerical mismatch
```

---

ktensor	<i>k-tensors</i>
---------	------------------

---

## Description

Functionality for  $k$ -tensors

## Usage

```
ktensor(S)
as.ktensor(M, coeffs)
is.ktensor(x)
## S3 method for class 'ktensor'
as.function(x, ...)
```

## Arguments

M, coeffs	Matrix of indices and coefficients, as in <code>spray(M, coeffs)</code>
S	Object of class <code>spray</code>
x	Object of class <code>ktensor</code>
...	Further arguments, currently ignored

## Details

A  $k$ -tensor object  $S$  is a map from  $V^k$  to the reals  $R$ , where  $V$  is a vector space (here  $R^n$ ) that satisfies multilinearity:

$$S(v_1, \dots, av_i, \dots, v_k) = a \cdot S(v_1, \dots, v_i, \dots, v_k)$$

and

$$S(v_1, \dots, v_i + v_i', \dots, v_k) = S(v_1, \dots, v_i, \dots, v_k) + S(v_1, \dots, v_i', \dots, v_k).$$

Note that this is *not* equivalent to linearity over  $V^{nk}$  (see examples).

In the **stokes** package,  $k$ -tensors are represented as sparse arrays (`spray` objects), but with a class of `c("ktensor", "spray")`. This is a natural and efficient representation for tensors that takes advantage of sparsity using **spray** package features.

Function `as.ktensor()` will coerce a  $k$ -form to a  $k$ -tensor via `kform_to_ktensor()`.

## Value

All functions documented here return a `ktensor` object except `as.function.ktensor()`, which returns a function.

## Author(s)

Robin K. S. Hankin

**References**

Spivak 1961

**See Also**

[tensorprod,kform,wedge](#)

**Examples**

```
as.ktensor(cbind(1:4,2:5,3:6),1:4)

## Test multilinearity:
k <- 4
n <- 5
u <- 3

## Define a randomish k-tensor:
S <- ktensor(spray(matrix(1+sample(u*k)%n,u,k),seq_len(u)))

## And a random point in V^k:
E <- matrix(rnorm(n*k),n,k)

E1 <- E2 <- E3 <- E

x1 <- rnorm(n)
x2 <- rnorm(n)
r1 <- rnorm(1)
r2 <- rnorm(1)

# change one column:
E1[,2] <- x1
E2[,2] <- x2
E3[,2] <- r1*x1 + r2*x2

f <- as.function(S)

r1*f(E1) + r2*f(E2) -f(E3) # should be small

## Note that multilinearity is different from linearity:
r1*f(E1) + r2*f(E2) - f(r1*E1 + r2*E2) # not small!
```

**Description**

Allows arithmetic operators to be used for  $k$ -forms and  $k$ -tensors such as addition, multiplication, etc, where defined.

**Usage**

```
## S3 method for class 'kform'
Ops(e1, e2 = NULL)
## S3 method for class 'ktensor'
Ops(e1, e2 = NULL)
```

**Arguments**

e1, e2                    Objects of class kform or ktensor

**Details**

The functions `Ops.kform()` and `Ops.ktensor()` pass unary and binary arithmetic operators (“+”, “-”, “\*”, “/” and “^”) to the appropriate specialist function by coercing to spray objects.

For wedge products of  $k$ -forms, use `wedge()` or `%^%` or `^`; and for tensor products of  $k$ -tensors, use `tensorprod()` or `%X%`.

**Value**

All functions documented here return an object of class kform or ktensor.

**Note**

A plain asterisk, “\*” behaves differently for ktensors and kforms. Given two ktensors T1, T2, then “T1\*T2” will return the their tensor product. This on the grounds that the idiom has only one natural interpretation. But its use is discouraged (use `%X%` or `tensorprod()` instead). An asterisk can also be used to multiply a tensor by a scalar, as in `T1*5`.

An asterisk cannot be used to multiply two kforms K1, K2, as in `K1*K2`, which will always return an error. This on the grounds that it has no sensible interpretation in general and you probably meant to use a wedge product, `K1^K2`. Note that multiplication by scalars is acceptable, as in `K1*6`. Further note that `K1*K2` returns an error even if one or both is a 0-form (or scalar), as in `K1*scalar(3)`. This behaviour may change in the future.

In the package the caret (“^”) evaluates the wedge product; note that `%^%` is also acceptable. Powers simply do not make sense for alternating forms: `S%^S = S^S` is zero identically. Here the caret is interpreted consistently as a wedge product, and if one of the factors is numeric it is interpreted as a zero-form (that is, a scalar). Thus `S^2 = wedge(S, 2) = 2^S = S*2 = S+S`, and indeed `S^n==S*n`. Caveat emptor! If `S` is a kform object, it is very tempting [but incorrect] to interpret “`S^3`” as something like “`S` to the power 3”. See also the note at `Ops.clifford` in the **clifford** package.

Powers are not implemented for ktensors on the grounds that a ktensor to the power zero is not defined.

Note that one has to take care with order of operations if we mix `^` with `*`. For example, `dx ^ (6*dy)` is perfectly acceptable; but `(dx ^ 6)*dy` will return an error, as will the unbracketed form `dx ^ 6 * dy`. In the second case we attempt to use an asterisk to multiply two k-forms, which triggers the error.

**Author(s)**

Robin K. S. Hankin



**Examples**

```
## dx_1 ^ dx_2 + 6dx_5 ^ dx_6:
as.kform(1) ^ as.kform(2) + 6*as.kform(5) ^ as.kform(6)

k1 <- kform_general(4,2,rnorm(6))
k2 <- kform_general(4,2,rnorm(6))

E <- matrix(rnorm(8),4,2)
as.function(k1+k2)(E)

## verify linearity, here 2*k1 + 3*k2:
as.function(2*k1+3*k2)(E)-(2*as.function(k1)(E) + 3*as.function(k2)(E))
## should be small
```

---

 phi

*Elementary tensors*


---

**Description**

Creates the elementary tensors or tensor products of elementary tensors

**Usage**

phi(n)

**Arguments**

n                      Vector of strictly non-negative integers

**Details**

If  $v_1, \dots, v_n$  is the standard basis for  $\mathbb{R}^n$  then  $\phi_i$  is defined so that  $\phi_i(v_j) = \delta_{ij}$ . phi(n) returns  $\phi_n$ .

If n is a vector of strictly positive integers, then phi(n) returns the tensor cross product of  $\phi$  applied to the individual elements of n [which is a lot easier and more obvious than it sounds].

**Note**

There is a vignette, phi

**Author(s)**

Robin K. S. Hankin

**Examples**

```
phi(6)
phi(6:8)

v <- sample(9)
phi(v) == Reduce("%X%", sapply(v, phi))
```

---

```
print.stokes          Print methods for k-tensors and k-forms
```

---

**Description**

Print methods for objects with options for printing in matrix form or multivariate polynomial form

**Usage**

```
## S3 method for class 'kform'
print(x, ...)
## S3 method for class 'ktensor'
print(x, ...)
```

**Arguments**

```
x          k-form or k-tensor
...        Further arguments (currently ignored)
```

**Details**

Printing is dispatched to `print.ktensor()` and `print.kform()` depending on its argument. Special dispensation is given for the zero object.

Although *k*-forms are alternating tensors and thus mathematically are tensors, they are handled differently.

The default print method uses the **spray** print methods, and as such respects the `polyform` option. However, setting `polyform` to `TRUE` can give misleading output, because `spray` objects are interpreted as multivariate polynomials not differential forms (and in particular uses the caret to signify powers).

It is much better to use options `ktensor_symbolic_print` or `kform_symbolic_print` instead: the bespoke print methods `print.kform()` and `print.ktensor()` are sensitive to these options.

For `kform` objects, if option `kform_symbolic_print` is non-null, the print method uses `as.symbolic()` to give an alternate way of displaying *k*-tensors and *k*-forms. The generic non-null value for this option would be “x” which gives output like “dx1 ^ dx2”. However, it has two special values: set `kform_symbolic_print` to “dx” for output like “dx ^ dz” and “txyz” for output like “dt ^ dx”, useful in relativistic physics with a Minkowski metric. See the examples.

For `ktensor` objects, if option `ktensor_symbolic_print` is `TRUE`, a different system is used. Given a tensor  $3\phi_4 \otimes \phi_1 - 5\phi_2 \otimes \phi_2$ , for example (where  $\phi_i(x^j) = \delta_i^j$ ), the method will give output that looks like “+3 d4\*d1 -5 d2\*d2”. I am not entirely happy with this and it might change in future.

More detail is given at `symbolic.Rd` and the `dx` vignette.

**Value**

Returns its argument invisibly.

**Note**

For both `kform` and `ktensor` objects, the `print` method asserts that its argument is a map from  $V^k$  to  $\mathbb{R}$  with  $V = \mathbb{R}^n$ . Here,  $n$  is the largest element in the index matrix. However, such a map naturally furnishes a map from  $(\mathbb{R}^m)^k$  to  $\mathbb{R}$ , provided that  $m \geq n$  via the natural projection from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . Formally this would be  $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0) \in \mathbb{R}^m$ . In the case of the zero  $k$ -form or  $k$ -tensor, “ $n$ ” is to be interpreted as “any  $n \geq 0$ ”. See also `dovs()`.

**Author(s)**

Robin K. S. Hankin

**See Also**

[as.symbolic.dovs](#)

**Examples**

```
a <- rform()
a

options(kform_symbolic_print = "x")
a

options(kform_symbolic_print = "dx")
kform(spray(kform_basis(3,2),1:3))

kform(spray(kform_basis(4,2),1:6)) # runs out of symbols

options(kform_symbolic_print = "txyz")
kform(spray(kform_basis(4,2),1:6)) # standard notation

options(kform_symbolic_print = NULL) # revert to default
a
```

---

rform

*Random kforms and ktensors*


---

**Description**

Random  $k$ -form objects and  $k$ -tensors, intended as quick “get you going” examples

**Usage**

```
rform(terms=9,k=3,n=7,coeffs,ensure=TRUE)
rtensor(terms=9,k=3,n=7,coeffs)
```

**Arguments**

terms	Number of distinct terms
k, n	A $k$ -form maps $V^k$ to $\mathbb{R}$ , where $V = \mathbb{R}^n$
coeffs	The coefficients of the form; if missing use <code>seq_len(terms)</code>
ensure	Boolean with default TRUE meaning to ensure that the <code>dovs()</code> of the returned value is in fact equal to <code>n</code> . If FALSE, sometimes the <code>dovs()</code> is strictly less than <code>n</code> because of random sampling

**Details**

Random  $k$ -form objects and  $k$ -tensors, of moderate complexity.

Note that argument `terms` is an upper bound, as the index matrix might contain repeats which are combined.

**Value**

All functions documented here return an object of class `kform` or `ktensor`.

**Author(s)**

Robin K. S. Hankin

**Examples**

```
(a <- rform())
(b <- rform())
a ^ b

a
a ^ dx
a ^ dx ^ dy

(x <- rtensor())
x %X% x
```

---

 scalar

*Scalars and losing attributes*


---

## Description

Scalars: 0-forms and 0-tensors

## Usage

```

scalar(s,kform=TRUE,lose=FALSE)
is.scalar(M)
`0form`(s=1,lose=FALSE)
`0tensor`(s=1,lose=FALSE)
## S3 method for class 'kform'
lose(M)
## S3 method for class 'ktensor'
lose(M)

```

## Arguments

<code>s</code>	A scalar value; a number
<code>kform</code>	Boolean with default TRUE meaning to return a kform and FALSE meaning to return a ktensor
<code>M</code>	Object of class ktensor or kform
<code>lose</code>	In function <code>scalar()</code> , Boolean with TRUE meaning to return a normal scalar, and default FALSE meaning to return a formal 0-form or 0-tensor

## Details

A  $k$ -tensor (including  $k$ -forms) maps  $k$  vectors to a scalar. If  $k = 0$ , then a 0-tensor maps no vectors to a scalar, that is, mapping nothing at all to a scalar, or what normal people would call a plain old scalar. Such forms are created by a couple of constructions in the package, specifically `scalar()`, `kform_general(1,0)` and `contract()`. These functions take a `lose` argument that behaves much like the `drop` argument in base extraction. Functions `0form()` and `0tensor()` are wrappers for `scalar()`.

Function `lose()` takes an object of class `ktensor` or `kform` and, if of arity zero, returns the coefficient.

Note that function `kform()` *always* returns a `kform` object, it never loses attributes.

There is a slight terminological problem. A  $k$ -form maps  $k$  vectors to the reals: so a 0-form maps 0 vectors to the reals. This is what anyone on the planet would call a scalar. Similarly, a 0-tensor maps 0 vectors to the reals, and so it too is a scalar. Mathematically, there is no difference between 0-forms and 0-tensors, but the package print methods make a distinction:

```

> scalar(5,kform=TRUE)
An alternating linear map from  $V^0$  to  $R$  with  $V=R^0$ :

```

```
    val
  =    5
> scalar(5,kform=FALSE)
A linear map from  $V^0$  to  $R$  with  $V=R^0$ :
    val
  =    5
>
```

Compare zero tensors and zero forms. A zero tensor maps  $V^k$  to the real number zero, and a zero form is an alternating tensor mapping  $V^k$  to zero (so a zero tensor is necessarily alternating). See `zero.Rd`.

### Value

The functions documented here return an object of class `kform` or `ktensor`, except for `is.scalar()`, which returns a Boolean.

### Author(s)

Robin K. S. Hankin

### See Also

[zeroform](#)

### Examples

```
o <- scalar(5)
o
lose(o)

kform_general(1,0)
kform_general(1,0,lose=FALSE)
```

### Description

A summary method for tensors and alternating forms, and a print method for summaries.

**Usage**

```
## S3 method for class 'kform'
summary(object, ...)
## S3 method for class 'ktensor'
summary(object, ...)
## S3 method for class 'summary.kform'
print(x, ...)
## S3 method for class 'summary.ktensor'
print(x, ...)
```

**Arguments**

```
object, x      Object of class ktensor or kform
...           Further arguments, passed to head()
```

**Details**

Summary methods for tensors and alternating forms. Uses `spray::summary()`.

**Author(s)**

Robin K. S. Hankin

**Examples**

```
a <- rform(100)
summary(a)
options(kform_symbolic_print = TRUE)
summary(a)
options(kform_symbolic_print = NULL) # restore default
```

---

symbolic

*Symbolic form*

---

**Description**

Returns a character string representing  $k$ -tensor and  $k$ -form objects in symbolic form. Used by the print method if either option `kform_symbolic_print` or `ktensor_symbolic_print` is non-null.

**Usage**

```
as.symbolic(M, symbols=letters, d="")
```

**Arguments**

M	Object of class kform or ktensor; a map from $V^k$ to $\mathbb{R}$ , where $V = \mathbb{R}^n$
symbols	A character vector giving the names of the symbols
d	String specifying the appearance of the differential operator

**Details**

Spivak (p89), in archetypically terse writing, states:

A function  $f$  is considered to be a 0-form and  $f \cdot \omega$  is also written  $f \wedge \omega$ . If  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable, then  $Df(p) \in \Lambda^1(\mathbb{R}^n)$ . By a minor modification we therefore obtain a 1-form  $df$ , defined by

$$df(p)(v_p) = Df(p)(v).$$

Let us consider in particular the 1-forms  $d\pi^i$ . It is customary to let  $x^i$  denote the *function*  $\pi^i$  (On  $\mathbb{R}^3$  we often denote  $x^1$ ,  $x^2$ , and  $x^3$  by  $x$ ,  $y$ , and  $z$ ). This standard notation has obvious disadvantages but it allows many classical results to be expressed by formulas of equally classical appearance. Since  $dx^i(p)(v_p) = d\pi^i(p)(v_p) = D\pi^i(p)(v) = v^i$ , we see that  $dx^1(p), \dots, dx^n(p)$  is just the dual basis to  $(e_1)_p, \dots, (e_n)_p$ . Thus every k-form  $\omega$  can be written

$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

Function `as.symbolic()` uses this format. For completeness, we add (p77) that  $k$ -tensors may be expressed in the form

$$\sum_{i_1, \dots, i_k=1}^n a_{i_1, \dots, i_k} \cdot \phi_{i_1} \otimes \dots \otimes \phi_{i_k}.$$

and this form is used for  $k$ -tensors. The print method for tensors, `print.ktensor()`, writes d1 for  $\phi_1$ , d2 for  $\phi_2$  [where  $\phi_i(x^j) = \delta_i^j$ ].

**Value**

Returns a “noquote” character string.

**Author(s)**

Robin K. S. Hankin

**See Also**

[print.stokes,dx](#)



**Examples**

```
(o <- kform_general(3,2,1:3))
as.symbolic(o,d="d",symbols=letters[23:26])
```

```
(a <- rform(n=50))
as.symbolic(a,symbols=state.abb)
```

---

tensorprod	<i>Tensor products of k-tensors</i>
------------	-------------------------------------

---

**Description**

Tensor products of  $k$ -tensors

**Usage**

```
tensorprod(U, ...)
tensorprod2(U1,U2)
```

**Arguments**

U, U1, U2	Object of class ktensor
...	Further arguments, currently ignored

**Details**

Given a  $k$ -tensor  $S$  and an  $l$ -tensor  $T$ , we can form the tensor product  $S \otimes T$ , defined as

$$S \otimes T(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) = S(v_1, \dots, v_k) \cdot T(v_{k+1}, \dots, v_{k+l}).$$

Package idiom for this includes `tensorprod(S,T)` and `S %X% T`; note that the tensor product is not commutative. Function `tensorprod()` can take any number of arguments (the result is well-defined because the tensor product is associative); it uses `tensorprod2()` as a low-level helper function.

**Value**

The functions documented here all return a spray object.

**Note**

The binary form `%X%` uses uppercase X to avoid clashing with `%x%` which is the Kronecker product in base R.

**Author(s)**

Robin K. S. Hankin

**References**

Spivak 1961

**See Also**

[ktensor](#)

**Examples**

```
(A <- ktensor(spray(matrix(c(1,1,2,2,3,3),2,3,byrow=TRUE),1:2)))
(B <- ktensor(spray(10+matrix(4:9,3,2),5:7)))
tensorprod(A,B)
```

```
A %X% B - B %X% A
```

```
Va <- matrix(rnorm(9),3,3)
Vb <- matrix(rnorm(38),19,2)
```

```
LHS <- as.function(A %X% B)(cbind(rbind(Va,matrix(0,19-3,3)),Vb))
RHS <- as.function(A)(Va) * as.function(B)(Vb)
```

```
c(LHS=LHS,RHS=RHS,diff=LHS-RHS)
```

---

transform

*Linear transforms of  $k$ -forms*

---

**Description**

Given a  $k$ -form, express it in terms of linear combinations of the  $dx_i$

**Usage**

```
pullback(K,M)
stretch(K,d)
```

**Arguments**

K	Object of class kform
M	Matrix of transformation
d	Numeric vector representing the diagonal elements of a diagonal matrix

**Details**

Function `pullback()` calculates the pullback of a function. A vignette is provided at ‘`pullback.Rmd`’.

Suppose we are given a two-form

$$\omega = \sum_{i < j} a_{ij} dx_i \wedge dx_j$$

and relationships

$$dx_i = \sum_r M_{ir} dy_r$$

then we would have

$$\omega = \sum_{i < j} a_{ij} \left( \sum_r M_{ir} dy_r \right) \wedge \left( \sum_r M_{jr} dy_r \right).$$

The general situation would be a  $k$ -form where we would have

$$\omega = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

giving

$$\omega = \sum_{i_1 < \dots < i_k} \left[ a_{i_1, \dots, i_k} \left( \sum_r M_{i_1 r} dy_r \right) \wedge \dots \wedge \left( \sum_r M_{i_k r} dy_r \right) \right].$$

The `transform()` function does all this but it is slow. I am not 100% sure that there isn’t a much more efficient way to do such a transformation. There are a few tests in `tests/testthat` and a discussion in the `stokes` vignette.

Function `stretch()` carries out the same operation but for  $M$  a diagonal matrix. It is much faster than `transform()`.

**Value**

The functions documented here return an object of class `kform`.

**Author(s)**

Robin K. S. Hankin

**References**

S. H. Weintraub 2019. *Differential forms: theory and practice*. Elsevier. (Chapter 3)

**See Also**

[wedge](#)

**Examples**

```

# Example in the text:
K <- as.kform(matrix(c(1,1,2,3),2,2),c(1,5))
M <- matrix(1:9,3,3)
pullback(K,M)

# Demonstrate that the result can be complicated:
M <- matrix(rnorm(25),5,5)
pullback(as.kform(1:2),M)

# Numerical verification:
o <- volume(3)

o2 <- pullback(pullback(o,M),solve(M))
max(abs(coeffs(o-o2))) # zero to numerical precision

# Following should be zero:
pullback(as.kform(1),M)-as.kform(matrix(1:5),c(crossprod(M,c(1,rep(0,4))))))

# Following should be TRUE:
issmall(pullback(o,crossprod(matrix(rnorm(10),2,5))))

# Some stretch() use-cases:

p <- rform()
p
stretch(p,seq_len(7))
stretch(p,c(1,0,0,1,1,1,1)) # kills dimensions 2 and 3

```

---

vector\_cross\_product    *The Vector cross product*

---

**Description**

The vector cross product  $\mathbf{u} \times \mathbf{v}$  for  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$  is defined in elementary school as

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

Function `vcp3()` is a convenience wrapper for this. However, the vector cross product may easily be generalized to a product of  $n-1$ -tuples of vectors in  $\mathbb{R}^n$ , given by package function `vector_cross_product()`.

Vignette `vector_cross_product`, supplied with the package, gives an extensive discussion of vector cross products, including formal definitions and verification of identities.

**Usage**

```

vector_cross_product(M)
vcp3(u,v)

```

**Arguments**

M Matrix with one more row than column; columns are interpreted as vectors  
 u, v Vectors of length 3, representing vectors in  $\mathbb{R}^3$

**Details**

A joint function profile for `vector_cross_product()` and `vcp3()` is given with the package at `vignette("vector_cross_product")`.

**Value**

Returns a vector

**Author(s)**

Robin K. S. Hankin

**See Also**

[wedge](#)

**Examples**

```
vector_cross_product(matrix(1:6,3,2))

M <- matrix(rnorm(30),6,5)
LHS <- hodge(as.1form(M[,1])^as.1form(M[,2])^as.1form(M[,3])^as.1form(M[,4])^as.1form(M[,5]))
RHS <- as.1form(vector_cross_product(M))
LHS-RHS # zero to numerical precision

# Alternatively:
hodge(Reduce(`^`, sapply(seq_len(5), function(i){as.1form(M[,i])}, simplify=FALSE)))
```

---

volume

*The volume element*

---

**Description**

The volume element in  $n$  dimensions

**Usage**

```
volume(n)
is.volume(K, n=dovs(K))
```

**Arguments**

n	Dimension of the space
K	Object of class kform

**Details**

Spivak phrases it well (theorem 4.6, page 82):

If  $V$  has dimension  $n$ , it follows that  $\Lambda^n(V)$  has dimension 1. Thus all alternating  $n$ -tensors on  $V$  are multiples of any non-zero one. Since the determinant is an example of such a member of  $\Lambda^n(V)$  it is not surprising to find it in the following theorem:

Let  $v_1, \dots, v_n$  be a basis for  $V$  and let  $\omega \in \Lambda^n(V)$ . If  $w_i = \sum_{j=1}^n a_{ij}v_j$  then

$$\omega(w_1, \dots, w_n) = \det(a_{ij}) \cdot \omega(v_1, \dots, v_n)$$

(see the examples for numerical verification of this).

Neither the zero  $k$ -form, nor scalars, are considered to be a volume element.

**Value**

Function `volume()` returns an object of class `kform`; function `is.volume()` returns a Boolean.

**Author(s)**

Robin K. S. Hankin

**References**

- M. Spivak 1971. *Calculus on manifolds*, Addison-Wesley

**See Also**

[zeroform](#), [as.1form](#), [dovs](#)

**Examples**

```
dx^dy^dz == volume(3)

p <- 1
for(i in 1:7){p <- p ^ as.kform(i)}
p
p == volume(7) # should be TRUE

o <- volume(5)
M <- matrix(runif(25),5,5)
det(M) - as.function(o)(M) # should be zero

is.volume(d(1) ^ d(2) ^ d(3) ^ d(4))
```

```
is.volume(d(1:9))
```

---

wedge	<i>Wedge products</i>
-------	-----------------------

---

### Description

Wedge products of  $k$ -forms

### Usage

```
wedge2(K1, K2)  
wedge(x, ...)
```

### Arguments

$K1, K2, x, \dots$   $k$ -forms

### Details

Wedge product of  $k$ -forms.

### Value

The functions documented here return an object of class `kform`.

### Note

In general use, use `wedge()` or `^` or `%^%`, as documented under `Ops`. Function `wedge()` uses low-level helper function `wedge2()`, which takes only two arguments.

A short vignette is provided with the package: `type vignette("wedge")` at the commandline.

### Author(s)

Robin K. S. Hankin

### See Also

[Ops](#)

### Examples

```
k1 <- as.kform(cbind(1:5,2:6),1:5)
k2 <- as.kform(cbind(5:7,6:8,7:9),1:3)
k3 <- kform_general(1:6,2)

a1 <- wedge2(k1,wedge2(k2,k3))
a2 <- wedge2(wedge2(k1,k2),k3)

is.zero(a1-a2) # NB terms of a1, a2 in a different order!

# This is why wedge(k1,k2,k3) is well-defined. Can also use ^:
k1 ^ k2 ^ k3
```

---

zap

*Zap small values in  $k$ -forms and  $k$ -tensors*

---

### Description

Equivalent to `zapsmall()`

### Usage

```
zap(X)
## S3 method for class 'kform'
zap(X)
## S3 method for class 'ktensor'
zap(X)
```

### Arguments

X                    Tensor or  $k$ -form to be zapped

### Details

Given an object of class `ktensor` or `kform`, coefficients close to zero are ‘zapped’, i.e., replaced by ‘0’, using `base::zapsmall()`.

Note, `zap()` actually changes the numeric value, it is not just a print method.

### Value

Returns an object of the same class

### Author(s)

Robin K. S. Hankin



**Examples**

```

S <- rform(7)
S == zap(S) # should be TRUE because the coeffs are integers

(a <- rform())
(b <- rform()*1e-11)
a+b
zap(a+b)

```

---

zero

*Zero tensors and zero forms*


---

**Description**

Correct idiom for generating zero  $k$ -tensors and  $k$ -forms

**Usage**

```

zeroform(n)
zerotensor(n)
is.zero(x)
is.empty(x)

```

**Arguments**

n	Arity of the $k$ -form or $k$ -tensor
x	Object to be tested for zero

**Value**

Returns an object of class `kform` or `ktensor`.

**Note**

Idiom such as `as.ktensor(rep(1, 5), 0)` and `as.kform(rep(1, 5), 0)` and indeed `as.kform(1:5, 0)` will return the zero tensor or  $k$ -form (in earlier versions of the package, these were held to be incorrect as the arity of the tensor was lost).

A 0-form is not the same thing as a zero tensor. A 0-form maps  $V^0$  to the reals; a scalar. A zero tensor maps  $V^k$  to zero. Some discussion is given at `scalar.Rd`.

**Author(s)**

Robin K. S. Hankin

**See Also**

[scalar](#)

**Examples**

```
zerotensor(5)
zeroform(3)
```

```
x <- rform(k=3)
x*0 == zeroform(3)      # should be true
x  == x + zeroform(3)  # should be true
```

```
y <- rtensor(k=3)
y*0 == zerotensor(3)   # should be true
y  == y+zerotensor(3) # should be true
```

```
## Following idiom is plausible but fails because as.ktensor(coeffs=0)
## and as.kform(coeffs=0) do not retain arity:
```

```
## as.ktensor(1+diag(5)) + as.ktensor(rep(1,5),0) # fails
## as.kform(matrix(1:6,2,3)) + as.kform(1:3,0) # also fails
```

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